



OPTIMAL CONSENSUS CONTROL OF DISCRETE-TIME STOCHASTIC MULTI-AGENT SYSTEMS

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ABSTRACT. This paper is concerned with the optimal consensus control of discrete-time multi-agent systems with multiplicative noise. The sufficient condition for the existence of a parameterized generalized algebraic Riccati equation (ARE) is firstly developed. Then, the sufficient condition on the control gain, the communication topology graph and the critical value of parameterized generalized ARE's parameter for mean square consensus are derived. Finally, the explicit control strategy is given to to guarantee consensus and minimize the performance index simultaneously.

1. Introduction. In the past twenty years, the consensus problem of multi-agent systems (MASs) has been paid much attention. In [22], Vicsek et al. demonstrated an interesting phenomenon: particles exhibit collective motion at high particle density and low localization noise. Jadbabaie et al. provided a theoretical explanation of the Vicsek model in [10]. Then, Olfati-Saber and Murray [16] considered first-order integrator dynamics and two control protocols were designed to solve the consensus problems for continuous-time and discrete-time systems, respectively. Ren [18] and He et al. [7] proposed consensus algorithms for second-order integrator dynamics and higher-order integrator dynamics, respectively. Different from previous unsigned graphs, Altafini [1] studied the consensus problem of signed graphs and showed that the system achieves bipartite consensus when the information exchange topology is structurally balanced. Additionally, Hu et al. [8] investigated the bipartite consensus of MASs with time delays in the presence of antagonistic interactions. Now, MASs have been widely used in various fields, such as sensor network, clustering of social insects, and unmanned aerial vehicles [32].

Note that the above significant progress addressed the consensus problem of perfect models, where each agent can obtain precise information from their neighbors. However, in practical applications of MASs, uncertain communication environments and measurement noises are inevitable [34]. Therefore, it is necessary to consider measurement noises when investigating multi-agent consensus problems. For MASs

2020 *Mathematics Subject Classification.* Primary: 49N10, 49J55, 49K45, 93E20.

Key words and phrases. Optimal consensus control, multi-agent systems, parameterized generalized algebraic Riccati equation, multiplicative noise.

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This paper is dedicated to the 70th birthday of Professor George Yin.

with multiplicative noise, Li et al. [15] gave some stochastic consensus conditions under connected graph and provided the upper and lower bounds for convergence rate. The consensus problem of continuous-time MASs in the presence of both communication latency and measurement noise was investigated in [35]. Zong et al. [33] gave the upper bound for delay terms to ensure p th moment exponentially stable. In [4] and [5], Djaidja et al. respectively studied the leader-following consensus of MASs with and without delay. For MASs driven by additive noise, in [13], Li and Zong derived some sufficient conditions to ensure stochastic weak group consensus, stochastic strong group consensus and hybrid group consensus, respectively. They further considered hybrid group consensus subjected to both communication latency and additive noise in [14]. In the presence of additive noise and time delay, Djaidja et al. [6] investigated the leader-following consensus. The previous work [36] established the consensus of both discrete-time first-order and second-order MASs with multiplicative noises. However, for discrete-time general linear stochastic MASs, little is known about the control design theory since the corresponding parameterized generalized algebraic Riccati equation (ARE) has not been well established.

Optimization problems of stochastic systems and MASs also attracted some attention. For single stochastic system, Zhang et al. [27] solved the optimal control problem of discrete-time stochastic systems with delay and measurement noise. Huang et al. [9] proposed an optimal controller for discrete-time systems driven by multiplicative noise based on an ARE. Wang et al. [24] investigated the optimal problem of single system involving by both state and control dependent multiplicative noise and input delay. For MASs, Movric and Lewis [12] examined the optimality of some distributed cooperative control protocols by a positive semi-definite quadratic performance criterion. Zhang et al. [31] studied the distributed optimal control of MASs with general linear dynamics. Jin et al. [11] investigated the distributed optimal consensus of stochastic continuous-time MASs with multiplicative noises. It is seemingly true that ARE for single discrete-time stochastic system can be used to design the optimal control of discrete-time stochastic MASs. However, it is worth noting that the decoupled subsystems of MASs depend on the eigenvalues of the Laplace matrix of communication graph. Thus, the classical ARE can not be used for stochastic MASs. This motivates us to develop optimal consensus control of discrete-time stochastic MASs with multiplicative noise by establishing a parameterized generalized ARE.

Motivated by the above discussion, this work studies the optimal consensus control of discrete-time MASs with multiplicative noise. Different from [27, 9], where the classical stochastic ARE was investigated, we firstly propose a parameterized generalized ARE with a parameter. By a recursive scheme, we establish the sufficient condition for the existence of positive definite solution to the parameterized generalized ARE. Moreover, the domain of the parameter is also obtained for parameterized generalized ARE to have the positive definite solution. Then, we give the control design of consensus control of MASs with multiplicative noises based on the relative measurements resorting to the parameterized generalized ARE. The explicit relationship between the control gain and the parameter in parameterized generalized ARE is revealed. Finally, the optimal consensus control strategy is developed to minimize a cost function based on the absolute state and relative measurements.

The rest of this paper is organized as follows. Section 2 gives the formulation of the optimal consensus problem and some preliminaries, mainly the algebraic graph theory. Section 3 investigates the sufficient condition for the parameterized generalized ARE to have a positive definite solution. Section 4 gives the mean square consensus of the MASs. Section 5 applies the consensus theory to obtain the optimal consensus control design. Some simulation results are presented in Section 6. Section 7 concludes the paper.

Notation: \mathbb{R}^n denotes n -dimensional column vectors. $\mathbb{R}^{n \times m}$ denotes the space of $n \times m$ real matrices. $\mathbf{1}_N$ stands for the n -dimensional unit column vector. I_n is the identify matrix. A' denotes the transpose of A . $M > 0$ stands for the symmetric matrix M is positive-definite. $M \geq 0$ stands for the symmetric matrix M is positive-semidefinite. $E[X]$ is the mathematical expectation of X . $\text{Range}(M)$ denotes the range of matrix M . \otimes denotes the Kronecker product. G^\dagger denotes the Moore-Penrose inverse [17, 20], satisfying $GG^\dagger G = G$, $G^\dagger GG^\dagger = G^\dagger$, $(GG^\dagger)' = GG^\dagger$ and $(G^\dagger G)' = G^\dagger G$.

2. Preliminaries and problem formulation.

2.1. Algebraic graph theory. We shall restrict our discussions mainly to the connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{M})$, where \mathcal{V} denotes the set of nodes and the adjacency matrix $\mathcal{M} = [a_{ij}] \in \mathbb{R}^{N \times N}$. If any two distinct agents of \mathcal{G} can be connected via a path, then we call an undirected graph \mathcal{G} connected. Also, $d_i = \sum_{j=1}^N a_{ij}$ is the degree of i . The Laplacian matrix is $\mathcal{L} = \mathcal{D} - \mathcal{M}$, where $\mathcal{D} = \text{diag}\{d_1, \dots, d_N\}$. We denote $0 = \lambda_1 \leq \dots \leq \lambda_N$ as the eigenvalues of L . It is well known that L always has a zero eigenvalue.

2.2. Problem formulation. Consider a MAS with N agents. The dynamics of agent i is:

$$x_i(k+1) = [Ax_i(k) + Bu_i(k)] + [Cx_i(k) + Du_i(k)]w(k), k = 0, 1, \dots, \quad (1)$$

where $i = 1, \dots, N$, $x_i(k) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times m}$, $u_i(k) \in \mathbb{R}^m$ is the input control of the i th agent and $\{w(k), k = 0, 1, 2, \dots\}$ is a sequence of real random variables defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k \geq 0}, \mathbb{P})$, which satisfies $E\{w(k)\} = 0$ and $E\{w^2(k)\} = 1$. Let $X(k) = [x_1'(k), \dots, x_N'(k)]'$. The cost function is given by

$$J_i = E \sum_{k=0}^{\infty} [x_i'(k)Qx_i(k) + u_i'(k)Ru_i(k)], \quad (2)$$

where Q and R are symmetric positive semidefinite matrices with appropriate dimensions. For the MAS (1), we define the admissible control set

$$\mathcal{U}_{ad} = \left\{ u(k) \mid u(k) \in \mathcal{L}_{\mathcal{F}}^2(\mathbb{R}^m) \text{ and } E \sum_{k=0}^{\infty} \|u(k)\|^2 < \infty \right\}.$$

The main objective of this paper is to find a distributed control protocol $u_i(k) \in \mathcal{U}_{ad}$ such that the MAS (1) achieve consensus and minimize the cost function $\sum_{i=1}^N J_i$. Because of the existence of measurement noise, the consensus is considered in mean square sense, which is defined as follows.

Definition 2.1. The MAS (1) is said to achieve mean square consensus if $\lim_{k \rightarrow \infty} E \|x_j(k) - x_i(k)\|^2 = 0, \forall i, j = 1, \dots, N$, for any given initial value $X(0)$.

3. A parameterized generalized algebraic Riccati equation. Algebraic Riccati equation is an important tool in control design for feedback stabilization of deterministic linear systems [2, 29] and stochastic systems [11, 23]. In this work, we also resort to a parameterized generalized ARE to design a control such that the mean square consensus is achieved and the cost function $\sum_{i=1}^N J_i$ is minimized.

In this section, we consider the following parameterized generalized ARE,

$$\begin{aligned} 0 &= A'\mathcal{P}A + C'\mathcal{P}C - \gamma(A'\mathcal{P}B + C'\mathcal{P}D) \\ &\quad \times (I_m + D'\mathcal{P}D + B'\mathcal{P}B)^{-1} (B'\mathcal{P}A + D'\mathcal{P}C) + I_n - \mathcal{P}, \gamma \in (0, 1). \end{aligned} \quad (3)$$

Firstly, we need to find the sufficient condition for the above parameterized generalized ARE to have a solution. Before establishing the existence of the solution \mathcal{P} , we consider the operator

$$\begin{aligned} H_\gamma(X) &= A'XA + C'XC + I_n \\ &\quad - \gamma(A'XB + C'XD)(I_m + B'XB + D'XD)^{-1}(B'XA + D'XC). \end{aligned} \quad (4)$$

We also define

$$\varphi(K, X) = (1 - \gamma)(A'XA + C'XC + I_n) + \gamma(M'_1XM_1 + M'_2XM_2 + I_n + K'K)$$

and

$$\Psi(K, X) = M'_1XM_1 + M'_2XM_2 + I_n + K'K,$$

where $M_1 = A + BK$, $M_2 = C + DK$. We obtain following lemmas.

Lemma 3.1. *The following statements are true.*

- 1) If $0 < \gamma_1 < \gamma_2 < 1$, then $H_{\gamma_1}(X) > H_{\gamma_2}(X)$.
- 2) If $X \geq Y$, then $H_\gamma(X) \geq H_\gamma(Y)$, $\gamma \in (0, 1)$.

Proof. 1) Note that $(A'XB + C'XD)(I_m + B'XB + D'XD)^{-1}(B'XA + D'XC) \geq 0$. Then,

$$\begin{aligned} H_{\gamma_1}(X) &= A'XA + C'XC + I_n \\ &\quad - \gamma_1(A'XB + C'XD)(I_m + B'XB + D'XD)^{-1}(B'XA + D'XC) \\ &\geq A'XA + C'XC + I_n \\ &\quad - \gamma_2(A'XB + C'XD)(I_m + B'XB + D'XD)^{-1}(B'XA + D'XC) \\ &= H_{\gamma_2}(X). \end{aligned}$$

- 2) If $X \geq Y$, then $H_\gamma(X) = \varphi(K_X, X) \geq \varphi(K_X, Y) \geq \varphi(K_Y, Y) = H_\gamma(Y)$. \square

Lemma 3.2. *The following statements 1-3 are equivalent.*

- 1) $\exists X > 0$ such that $X > H_\gamma(X)$.
- 2) $\exists K, X > 0$ such that $X > \varphi(K, X)$.
- 3) $\exists V$ and $0 \leq W \leq I$ such that the following LMI holds:

$$\Omega_\gamma(W, V) = \begin{bmatrix} W & \sqrt{\gamma}E' & \sqrt{\gamma}F' & \sqrt{1-\gamma}WA' & \sqrt{1-\gamma}WC' \\ \sqrt{\gamma}E & W & & & \\ \sqrt{\gamma}F & & W & & \\ \sqrt{1-\gamma}AW & & & W & \\ \sqrt{1-\gamma}CW & & & & W \end{bmatrix} > 0, \quad (5)$$

where $\gamma \in (0, 1)$, $E = AW + BV$ and $F = CW + DV$.

Proof. 1) \Rightarrow 2). Letting $K = K_X = -(I_m + D'XD + B'XB)^{-1} (B'XA + D'XC)$, we can easily obtain $X > H_\gamma(X) = \varphi(K, X)$.

2) \Rightarrow 1). Note that $\operatorname{argmin}_K \varphi(K, X) = \operatorname{argmin}_K \Psi(K, X)$. Since $X \geq 0$, it follows that $\Psi(K, X)$ is quadratic and convex in the variable K . Therefore, we can obtain the minimum value by solving $\frac{\partial \Psi(K, X)}{\partial K} = 0$ and one easily finds $K_X = -(I_m + D'XD + B'XB)^{-1} (B'XA + D'XC)$. Then, we have $X > \varphi(K, X) \geq \min_K \varphi(K, X) = \varphi(K_X, X) = H_\gamma(X)$ for any K .

2) \Leftrightarrow 3). From $X > \varphi(K, X) = (1 - \gamma)(A'XA + C'XC + I_n) + \gamma(M_1'XM_1 + M_2'XM_2 + I_n + K'K)$, we obtain $X - (1 - \gamma)(A'XA + C'XC) > \gamma(M_1'XM_1 + M_2'XM_2 + K'K) + I_n \geq I_n > 0$. Using the Schur complement decomposition, we have

$$\theta = \begin{bmatrix} X & \sqrt{\gamma}M_1' & \sqrt{\gamma}M_2' & \sqrt{1-\gamma}A' & \sqrt{1-\gamma}C' \\ \sqrt{\gamma}M_1 & X^{-1} & & & \\ \sqrt{\gamma}M_2 & & X^{-1} & & \\ \sqrt{1-\gamma}A & & & X^{-1} & \\ \sqrt{1-\gamma}C & & & & X^{-1} \end{bmatrix} > 0.$$

This is equivalent to

$$\begin{bmatrix} X^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \theta \begin{bmatrix} X^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \\ = \begin{bmatrix} X^{-1}\sqrt{\gamma}X^{-1}M_1' & \sqrt{\gamma}X^{-1}M_2' & \sqrt{1-\gamma}X^{-1}A' & \sqrt{1-\gamma}X^{-1}C' \\ \sqrt{\gamma}M_1X^{-1} & X^{-1} & & \\ \sqrt{\gamma}M_2X^{-1} & & X^{-1} & \\ \sqrt{1-\gamma}AX^{-1} & & & X^{-1} \\ \sqrt{1-\gamma}CX^{-1} & & & & X^{-1} \end{bmatrix} > 0.$$

Letting $W = X^{-1} > 0$, $V = KW^{-1}$, $E = AW + BV$ and $F = CW + DV$, the previous LMI is equivalent to

$$\Omega_\gamma(W, V) = \begin{bmatrix} W & \sqrt{\gamma}E' & \sqrt{\gamma}F' & \sqrt{1-\gamma}WA' & \sqrt{1-\gamma}WC' \\ \sqrt{\gamma}E & W & & & \\ \sqrt{\gamma}F & & W & & \\ \sqrt{1-\gamma}AW & & & W & \\ \sqrt{1-\gamma}CW & & & & W \end{bmatrix} > 0.$$

Since $\Omega_\gamma(\alpha W, \alpha V) = \alpha \Omega_\gamma(W, V)$, W can be restricted to $W \leq I$. The proof is complete. \square

Lemma 3.3. Define $L(X) = (1 - \gamma)(A'XA + C'XC) + \gamma(M_1'XM_1 + M_2'XM_2)$, where $M_1 = A + BK$, $M_2 = C + DK$. Then, if there exists a matrix $\bar{X} > 0$ such that $\bar{X} > L(\bar{X})$, the following statements are true.

- a): For all $V \geq 0$, $\lim_{k \rightarrow \infty} L^k(V) = 0$.
- b): The sequence $Z_{k+1} = L(Z_k) + U$ is bounded for all $U \geq 0$ and any initial value $Z_0 \geq 0$.

Proof. It can be seen that $L(X) \geq 0$ for all $X \geq 0$. Also, if $X \geq Y$, we have $L(X) \geq L(Y)$. For given $V \geq 0$, we choose $r_V \in [0, 1)$ and $b_V \geq 0$ such that

$L(\bar{X}) < r_V \bar{X}$ and $b_V \bar{X} \geq V$. Then, it can be shown that

$$0 \leq L^k(V) \leq b_V L^k(\bar{X}) < b_V r_V^k \bar{X},$$

which implies $\lim_{k \rightarrow \infty} L^k(V) = 0$, that is, the statement **a**) holds. Also, we have

$$\begin{aligned} Z_k &= L^k(Z_0) + \sum_{t=0}^{k-1} L^t(U) \\ &\leq \left(b_{Z_0} r^k + \sum_{t=0}^{k-1} b_U r_{Z_0}^t \right) \bar{X} \\ &\leq \left(b_{Z_0} r_{Z_0}^k + \frac{b_U}{1-r_{Z_0}} \right) \bar{X} \\ &\leq \left(b_{Z_0} + \frac{b_U}{1-r_{Z_0}} \right) \bar{X}, \end{aligned}$$

which implies the statement **b**). \square

Based on the lemmas above, we now establish the existence of the positive solution to parameterized generalized ARE (3) by constructing the corresponding recursive scheme,

$$\begin{aligned} \mathcal{P}_{t+1} &= A' \mathcal{P}_t A + C' \mathcal{P}_t C + I_n - \gamma (A' \mathcal{P}_t B + C' \mathcal{P}_t D) \\ &\quad \times (I_m + B' \mathcal{P}_t B + D' \mathcal{P}_t D)^{-1} (B' \mathcal{P}_t A + D' \mathcal{P}_t C), t = 0, 1, 2, \dots \end{aligned} \quad (6)$$

Then, we have $P_{t+1} = H_\gamma(P_t) = H_\gamma^{t+1}(P_0)$ from the definition of $H_\gamma(\cdot)$, where $H_\gamma^{t+1}(P_0)$ represents the value after $t+1$ iterations from P_0 .

Theorem 3.4. *If there exist matrices \bar{K} and $\bar{X} > 0$ such that $\bar{X} > \varphi(\bar{K}, \bar{X})$, then the Riccati iteration (6) converges for any initial condition $\mathcal{P}_0 \geq 0$, and the limit is independent of \mathcal{P}_0 , denoted by $\lim_{t \rightarrow \infty} \mathcal{P}_t = \lim_{t \rightarrow \infty} H_\gamma^t(\mathcal{P}_0) = \bar{\mathcal{P}}$, i.e., the parameterized generalized ARE (3) admits a solution.*

Proof. We first prove that $P_t = H_\gamma^t(P_0)$ is bounded for any $P_0 \geq 0$, i.e., $P_t \leq S_{P_0}$ for certain $S_{P_0} > 0$, if there exist matrices \bar{K} and $\bar{X} > 0$ such that $\bar{X} > \varphi(\bar{K}, \bar{X})$. Let $\bar{L}(X) = (1 - \gamma)(A' X A + C' X C) + \gamma(M_1' X M_1 + M_2' X M_2)$, where $M_1 = A + B\bar{K}$, $M_2 = C + D\bar{K}$. We can see that

$$\bar{X} > \varphi(\bar{K}, \bar{X}) = \bar{L}(\bar{X}) + I_n + \gamma \bar{K}' \bar{K} > \bar{L}(\bar{X}).$$

Consequently, L satisfies the condition of Lemma 3.3. We have

$$P_{t+1} = H_\gamma(P_t) \leq \varphi(\bar{K}, P_t) = L(P_t) + I_n + \gamma \bar{K}' \bar{K} =: L(P_t) + U,$$

where $U = I_n + \gamma \bar{K}' \bar{K} > 0$. From Lemma 3.3, P_t is bounded.

Now, we show that the Riccati iteration (6) converges for any initial condition $\mathcal{P}_0 \geq 0$ by three steps.

Firstly, we initialize the Riccati iteration (6) at $U_0 = 0$, i.e., $U_k = H_\gamma^k(0)$. One can see that $0 = U_0 < U_1 = I_n$ and $U_1 = H_\gamma(U_0) < H_\gamma(U_1) = U_2$ from Lemma 3.1. Then, we have $0 = U_0 < U_1 < U_2 < \dots < S_{U_0}$. One can see that the sequence converges and the limit is denoted by $\lim_{k \rightarrow \infty} U_k = \bar{P}$, that is, the fixed point \bar{P} satisfies $\bar{P} = H_\gamma(\bar{P})$.

Then, we initialize the Riccati iteration (6) at $T_0 \geq \bar{P}$. Define $\tilde{L}(T) = (1 - \gamma)(A'TA + C'TC) + \gamma(M_1'TM_1 + M_2'TM_2)$, where $M_1 = A + BK_{\bar{P}}$, $M_2 = C + DK_{\bar{P}}$ and $K_{\bar{P}} = -(I_m + D'\bar{P}D + B'\bar{P}B)^{-1}(B'\bar{P}A + D'\bar{P}C)$. Then, we have

$$\bar{P} = H_\gamma(\bar{P}) = \tilde{L}(\bar{P}) + I_n + \gamma K_{\bar{P}}' K_{\bar{P}} > \tilde{L}(\bar{P}).$$

Consequently, \tilde{L} satisfies the condition of Lemma 3.3 and we have $\lim_{k \rightarrow \infty} \tilde{L}^k(T) = 0$, $\forall T \geq 0$. One can see that $T_1 = H_\gamma(T_0) \geq H_\gamma(\bar{P}) = \bar{P}$. Further we obtain $T_k = H_\gamma^k(T_0) \geq \bar{P}$, $\forall k > 0$. Observe that

$$\begin{aligned} 0 &\leq (T_{k+1} - \bar{P}) = H_\gamma(T_k) - H_\gamma(\bar{P}) \\ &= \varphi(K_{T_k}, T_k) - \varphi(K_{\bar{P}}, \bar{P}) \leq \varphi(K_{\bar{P}}, T_k) - \varphi(K_{\bar{P}}, \bar{P}) \\ &= (1 - \gamma) [A'(T_k - \bar{P})A + C'(T_k - \bar{P})C] \\ &\quad + \gamma [M_1'(T_k - \bar{P})M_1 + M_2'(T_k - \bar{P})M_2] \\ &= \tilde{L}(T_k - \bar{P}). \end{aligned}$$

Thus, we have $0 \leq \lim_{k \rightarrow \infty} (T_{k+1} - \bar{P}) \leq \lim_{k \rightarrow \infty} \tilde{L}(T_k - \bar{P}) = 0$, which implies that the sequence $H_\gamma^k(T_0)$ also converges when initializing at $T_0 \geq \bar{P}$.

Finally, we initialize the Riccati iteration (6) at any given $P_0 \geq 0$. Denote $U_0 = 0$, $T_0 = P_0 + \bar{P}$ and $U_0 \leq P_0 \leq T_0$. Then, we have $H_\gamma(U_0) \leq H_\gamma(P_0) \leq H_\gamma(T_0)$ from Lemma 3.1, i.e., $U_1 \leq P_1 \leq T_1$. Further we obtain $U_k \leq P_k \leq T_k$, $\forall k \geq 0$. Therefore, $\lim_{k \rightarrow \infty} P_k = \bar{P}$. The proof is complete. \square

Theorem 3.5. *Assume that $\gamma^* = \operatorname{argmin}_\gamma [\exists \hat{P} > 0 \mid \hat{P} > H_\gamma(\hat{P})] \in (0, 1)$ is well defined. Then, the parameterized generalized ARE (3) admits a solution $\mathcal{P} > 0$ for any $\gamma \in (\gamma^*, 1)$.*

Proof. When $\gamma = \gamma^*$, there exists a matrix $\hat{P} > 0$ such that $\hat{P} > H_{\gamma^*}(\hat{P})$. From Lemma 3.2, there exists matrices $K, X > 0$ such that $X > \varphi(K, X)$. Therefore, we obtain that $\mathcal{P}_{t+1} = H_\gamma(\mathcal{P}_t)$ converges from Theorem 3.4, that is, the parameterized generalized ARE (3) admits a solution. If $\gamma = \gamma_2 \in (\gamma^*, 1)$, then $H_{\gamma^*}(\hat{P}) > H_{\gamma_2}(\hat{P})$ from Lemma 3.1 and we have $\hat{P} > H_{\gamma^*}(\hat{P}) > H_{\gamma_2}(\hat{P})$. Similar to the case of $\gamma = \gamma^*$, we obtain that the parameterized generalized ARE (3) admits a solution, which completes the proof. \square

Remark 3.6. Combining Theorem 3.5 and Lemma 3.2, γ^* can be numerically computed by $\gamma^* = \operatorname{argmin}_\gamma \{\Omega_\gamma(W, V) > 0, 0 \leq W \leq I\}$, where $\Omega_\gamma(W, V)$ is defined in (5).

Remark 3.7. Note that Theorem 3.5 requires $\gamma^* \in (0, 1)$ to be well defined. This together with Theorem 3.4 implies the existence of the positive definite solution P . In fact, the existence condition $\gamma^* \in (0, 1)$ depends on the coefficient matrices A, B, C, D . To see it clearly, we consider the case $B = D, C = 0$, that is, the stochastic system has the following form

$$x(k+1) = Ax(k) + Bu(k) + Du(k)w(k). \quad (7)$$

It is not difficult to obtain its ARE

$$P = A'PA + I_n - \frac{1}{2}A'PB(I_m + B'PB)^{-1}B'PA, \gamma = \frac{1}{2}. \quad (8)$$

It is proved in [19, Lemma 5.4] for unstable A that if (A, B) is controllable and $\Pi_i |\lambda_i(A)|^u < 2$, then (8) has the unique $P > 0$, where $\lambda_i^u(A)$ are the unstable

eigenvalues of A . Without the two conditions, (8) may not admit the positive definite solution for $\gamma \in (0, 1)$, and then γ^* is unavailable.

4. Mean square consensus. In this section, we will derive the condition for mean square consensus so as to solve the optimal consensus control problem in the next section.

We use the classical feedback control protocol

$$u_i(k) = \bar{K} \sum_{j=1}^N a_{ij} (x_j(k) - x_i(k)), \quad (9)$$

where $\bar{K} \in \mathbb{R}^{m \times n}$ is the gain matrix to be designed.

Applying Theorem 3.5 produces the following theorem, which gives sufficient condition for MAS (1) under (9) to achieve mean square consensus.

Theorem 4.1. *If $\gamma^* = \operatorname{argmin}_\gamma [\exists \hat{\mathcal{P}} > 0 \mid \hat{\mathcal{P}} > H_\gamma(\hat{\mathcal{P}})] \in (0, 1)$ is well defined, then for any $1 > \gamma \geq \gamma^*$, mean square consensus can be achieved under the control protocol (9) with consensus gain*

$$\bar{K} = k(I_m + D'PD + B'PB)^{-1}(B'PA + D'PC) \quad (10)$$

where

$$-\frac{\sqrt{1-\gamma}}{\lambda_2} + \frac{1}{\lambda_2} \leq k \leq \frac{\sqrt{1-\gamma}}{\lambda_N} + \frac{1}{\lambda_N}, \quad (11)$$

\mathcal{P} satisfies the parameterized generalized ARE (3).

Proof. With the protocol (9), the closed-loop subsystem takes the form

$$\begin{aligned} x_i(k+1) &= \begin{bmatrix} Ax_i(k) + B\bar{K} \sum_{j=1}^N a_{ij} (x_j(k) - x_i(k)) \\ Cx_i(k) + D\bar{K} \sum_{j=1}^N a_{ij} (x_j(k) - x_i(k)) \end{bmatrix} \\ &\quad + \begin{bmatrix} \\ \\ \end{bmatrix} w(k). \end{aligned} \quad (12)$$

We can rewrite the above equation as

$$X(k+1) = [I_N \otimes A - \mathcal{L} \otimes (B\bar{K})] X(k) + [I_N \otimes C - \mathcal{L} \otimes (D\bar{K})] X(k)w(k).$$

Denote consensus error

$$\delta_i(k) = x_i(k) - \frac{1}{N} \sum_{m=1}^N x_m(k), \quad (i = 1, 2, \dots, N)$$

and consensus error vector $\delta(k) = [\delta_1'(k), \dots, \delta_N'(k)]'$. We have

$$\delta(k) = \left[\left(I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}'_N \right) \otimes I_n \right] X(k),$$

which implies

$$\delta(k+1) = [I_N \otimes A - \mathcal{L} \otimes (B\bar{K})] \delta(k) + [I_N \otimes C - \mathcal{L} \otimes (D\bar{K})] \delta(k)w(k).$$

Define the unitary matrix $\Psi = \left[\frac{\mathbf{1}_N}{\sqrt{N}}, \phi_2, \dots, \phi_N \right]$, where ϕ_i is the unit eigenvector of \mathcal{L} associated with the eigenvalue λ_i , i.e., $\phi_i' \mathcal{L} = \lambda_i \phi_i'$, $\phi_i \in \mathbb{R}^N$. \mathcal{L} can be transformed

into a diagonal form $\Psi' \mathcal{L} \Psi = \text{diag} \{0, \lambda_2, \dots, \lambda_N\}$. Denote $\delta(k) = (\Psi \otimes I_n) \tilde{\delta}(k)$ and $\tilde{\delta}(k) = [\tilde{\delta}'_1(k), \dots, \tilde{\delta}'_N(k)]'$. Then, we have $\tilde{\delta}_1(k) = 0$ and

$$\begin{aligned} \tilde{\delta}(k+1) &= (\Psi \otimes I_n)' \delta(k+1) \\ &= [I_N \otimes A - \text{diag} \{0, \lambda_2 B \bar{K}, \dots, \lambda_N B \bar{K}\}] \tilde{\delta}(k) \\ &\quad + [I_N \otimes C - \text{diag} \{0, \lambda_2 D \bar{K}, \dots, \lambda_N D \bar{K}\}] \tilde{\delta}(k) w(k). \end{aligned}$$

Letting $\xi(k) = [\tilde{\delta}'_2(k), \dots, \tilde{\delta}'_N(k)]'$, we have

$$\xi(k+1) = R_1 \xi(k) + M_1 \xi(k) w(k), \quad (13)$$

where $R_1 = I_{N-1} \otimes A - \wedge \otimes B \bar{K}$, $M_1 = I_{N-1} \otimes C - \wedge \otimes D \bar{K}$, $\wedge = \text{diag} \{\lambda_2, \dots, \lambda_N\}$.

Consider the following Lyapunov function

$$V(k) = \xi'(k) (I_{N-1} \otimes \mathcal{P}) \xi(k), \quad (14)$$

where \mathcal{P} satisfies the parameterized generalized ARE (3). Substituting (13) into (14) and taking the expectation on both sides of (14), we have

$$EV(k+1) = E \{ \xi'(k) [R'_1 (I_{N-1} \otimes \mathcal{P}) R_1 + M'_1 (I_{N-1} \otimes \mathcal{P}) M_1] \xi(k) \}, \quad (15)$$

which implies

$$\begin{aligned} EV(k+1) - EV(k) &= E \{ \xi'(k) [R'_1 (I_{N-1} \otimes \mathcal{P}) R_1 \\ &\quad + M'_1 (I_{N-1} \otimes \mathcal{P}) M_1 - I_{N-1} \otimes \mathcal{P}] \xi(k) \}. \end{aligned}$$

We multiply both sides by β^{k+1} , where $\beta > 1$. Then, we have

$$\beta^{k+1} EV(k+1) - \beta^k EV(k) \leq (\beta^{k+1} - \beta^k) EV(k) + \beta^{k+1} E \xi'(k) R_2 \xi(k),$$

where $R_2 := R'_1 (I_{N-1} \otimes \mathcal{P}) R_1 + M'_1 (I_{N-1} \otimes \mathcal{P}) M_1 - I_{N-1} \otimes \mathcal{P}$. Then, we can rewrite the above formula as

$$\beta^{k+1} EV(k+1) \leq EV(0) + \sum_{s=0}^k (\beta^{s+1} - \beta^s) EV(s) + \sum_{s=0}^k \beta^{s+1} E \xi'(s) R_2 \xi(s).$$

It can be seen that $EV(s) \leq \|\mathcal{P}\| \|\xi(s)\|^2$. Thus,

$$\begin{aligned} \beta^{k+1} EV(k+1) &\leq EV(0) + (1 - \beta^{-1}) \|\mathcal{P}\| \sum_{s=0}^k \beta^{s+1} \|\xi(s)\|^2 + \sum_{s=0}^k \beta^{s+1} E \xi'(s) R_2 \xi(s) \\ &\leq EV(0) + \sum_{s=0}^k \beta^{s+1} E \xi'(s) R_3 \xi(s), \end{aligned} \quad (16)$$

where $R_3 = R_2 + (1 - \beta^{-1}) \|\mathcal{P}\| I_{N-1}$.

Note that $(1 - \beta^{-1}) \|\mathcal{P}\| I_{N-1} > 0$. Then, there exists $\beta > 1$ such that $R_3 < 0$ if $R_2 < 0$. In this case, we can obtain from (16) that

$$\beta^{k+1} EV(k+1) \leq EV(0).$$

We have $\lambda_{\min}(\mathcal{P}) \|\xi(k+1)\|^2 \leq V(k+1)$, it follows that

$$\lambda_{\min}(\mathcal{P}) \beta^{k+1} E \|\xi(k+1)\|^2 \leq \beta^{k+1} EV(k+1) \leq EV(0).$$

Then, we have

$$E \|\xi(k+1)\|^2 \leq \frac{EV(0)}{\lambda_{\min}(\mathcal{P}) \beta^{k+1}}. \quad (17)$$

Hence, one obtains that

$$\lim_{k \rightarrow \infty} E \|\xi(k+1)\|^2 = 0.$$

This combining with the definition of $\xi(k)$ and $\delta(k)$ implies

$$\lim_{k \rightarrow \infty} E \left\| x_j(k) - \frac{1}{N} \sum_{m=1}^N x_m(k) \right\|^2 = 0. \quad (18)$$

Therefore, mean square consensus will follow if $R_2 < 0$.

Next, we prove that $\bar{K} = k(I_m + D'\mathcal{P}D + B'\mathcal{P}B)^{-1}(B'\mathcal{P}A + D'\mathcal{P}C)$ under condition (11) can assure $R_2 < 0$. It can be seen that R_2 can be reformulated as

$$\begin{aligned} R_2 = & \text{diag} \left\{ (A - \lambda_2 B \bar{K})' \mathcal{P} (A - \lambda_2 B \bar{K}), \dots, \right. \\ & \left. (A - \lambda_N B \bar{K})' \mathcal{P} (A - \lambda_N B \bar{K}) \right\} \\ & + \text{diag} \left\{ (C - \lambda_2 D \bar{K})' \mathcal{P} (C - \lambda_2 D \bar{K}) - \mathcal{P}, \dots, \right. \\ & \left. (C - \lambda_N D \bar{K})' \mathcal{P} (C - \lambda_N D \bar{K}) - \mathcal{P} \right\}. \end{aligned}$$

This is equivalent to

$$(A - \lambda_i B \bar{K})' \mathcal{P} (A - \lambda_i B \bar{K}) + (C - \lambda_i D \bar{K})' \mathcal{P} (C - \lambda_i D \bar{K}) - \mathcal{P} < 0, \quad (19)$$

where $i = 2, \dots, N$. Substituting $\bar{K} = k(I_m + D'\mathcal{P}D + B'\mathcal{P}B)^{-1}(B'\mathcal{P}A + D'\mathcal{P}C)$ into (19), we can obtain that (19) can be ensured if

$$\begin{aligned} & A'\mathcal{P}A + C'\mathcal{P}C - (2\lambda_i k - \lambda_i^2 k^2)(A'\mathcal{P}B + C'\mathcal{P}D) \\ & \times (I_m + D'\mathcal{P}D + B'\mathcal{P}B)^{-1}(B'\mathcal{P}A + D'\mathcal{P}C) - \mathcal{P} < 0, \end{aligned} \quad (20)$$

where $i = 2, \dots, N$. Note that (11) implies (20). Now, the proof is completed. \square

Remark 4.2. From [25], we know that the mean square exponential stability of the MAS (1) implies the almost sure exponential stability. Moreover, (17) implies mean square exponential stability of the MAS (13). Hence, the MAS (1) can achieve both mean square and almost sure consensus simultaneously under (10) and (11).

5. Optimal consensus. Based on the consensus result presented in the last section, we are going to provide the optimal consensus control strategy to guarantee mean square consensus and minimize the performance index simultaneously.

Firstly, define the following ARE

$$0 = A'PA + Q - P + C'PC - H'G^\dagger H, \quad (21)$$

where $H = B'PA + D'PC$, $G = R + B'PB + D'PD$. For the convenience of the following derivation, we denote

$$A = A - BG^\dagger H, B = B(I - G^\dagger G), C = C - DG^\dagger H, D = D(I - G^\dagger G).$$

Here, we assume that $\text{Range}(H) \subseteq \text{Range}(G)$, i.e. regular case. For the optimal control, we also define

$$\begin{aligned} \mathcal{H}_\gamma(X) &= A'XA + C'XC + I_n \\ &- \gamma(A'XB + C'XD)(I_m + B'XB + D'XD)^{-1}(B'XA + D'XC). \end{aligned}$$

The following theorem gives the optimal consensus control strategy.

Theorem 5.1. *Assume that (21) has the solution $P \geq 0$, and $\gamma^* = \text{argmin}_\gamma[\exists \hat{\mathcal{P}} > 0 \mid \hat{\mathcal{P}} > \mathcal{H}_\gamma(\hat{\mathcal{P}})] \in (0, 1)$ is well defined. Then, for any $\gamma \geq \gamma^*$, the optimal consensus solution to minimize (2) is*

$$u_i(k) = -G^\dagger H x_i(k) + (I - G^\dagger G) z_i(k), \quad (22)$$

where

$$z_i(k) = K \sum_{j=1}^N a_{ij} (x_j(k) - x_i(k)),$$

$$K = k (I_m + D'PD + B'PB)^{-1} (B'PA + D'PC),$$

k satisfies (11), and \mathcal{P} obeys $\mathcal{P} = \mathcal{H}_\gamma(\mathcal{P})$.

Proof. Note that

$$\begin{aligned} & E \left\{ \sum_{k=0}^{\infty} (x'_i(k+1)Px_i(k+1) - x'_i(k)Px_i(k)) \right\} \\ &= E \left\{ \lim_{k \rightarrow \infty} x_i(k+1)'Px_i(k+1) - x_i(0)'Px_i(0) \right\}, \end{aligned}$$

and

$$\begin{aligned} E[x'_i(k+1)Px_i(k+1)] &= E[(Ax_i(k) + Bu_i(k) + Cx_i(k)w_k + Du_i(k)w_k)' \\ &\quad P(Ax_i(k) + Bu_i(k) + Cx_i(k)w_k + Du_i(k)w_k)] \\ &= E[x'_i(k)[A'PA + C'PC]x_i(k) \\ &\quad + x'_i(k)[A'PB + C'PD]u'_i(k) \\ &\quad + u'_i(k)[B'PB + D'PD]u_i(k) \\ &\quad + u'_i(k)[B'PA + D'PC]x'_i(k)]. \end{aligned}$$

Substituting the above equalities into the cost function

$$J_i = E \sum_{k=0}^{\infty} [x'_k(k)Qx_i(k) + u'_k(k)Ru_k(k)]$$

and using the MAS (1), we have

$$\begin{aligned} \sum_{i=1}^N J_i &= \sum_{i=1}^N E \left(\sum_{k=0}^{\infty} [x'_i(k)Qx_i(k) + u'_i(k)Ru_i(k) + x'_i(k+1)Px_i(k+1) \right. \\ &\quad \left. - x'_i(k)Px_i(k) + \sum_{i=1}^N E(x'_i(0)Px_i(0)) - \sum_{i=1}^N E(\lim_{k \rightarrow \infty} x'_i(k)Px_i(k)) \right) \\ &= \sum_{i=1}^N \sum_{k=0}^{\infty} E \{ x'_i(k)[Q - P + A'PA + C'PC]x_i(k) \\ &\quad + x'_i(k)[A'PB + C'PD]u_i(k) + u'_i(k)[B'PB + D'PD + R]u_i(k) \\ &\quad + u'_i(k)[B'PA + D'PC]x_i(k) \} + \sum_{i=1}^N E \{ x'_i(0)Px_i(0) \} \\ &\quad - \sum_{i=1}^N E \left\{ \lim_{k \rightarrow \infty} x'_i(k)Px_i(k) \right\}. \end{aligned}$$

Denote $M_1 = \sum_{i=1}^N E \{ x'_i(0)Px_i(0) \}$ and $M_2 = \sum_{i=1}^N E \{ \lim_{k \rightarrow \infty} x'_i(k)Px_i(k) \}$.

Applying ARE (21), we have

$$\sum_{i=1}^N J_i = \sum_{i=1}^N \sum_{k=0}^{\infty} E \{ x'_i(k)H'G^\dagger Hx_i(k) + x'_i(k)H'u_i(k) \}$$

$$\begin{aligned}
& +u'_i(k)Gu_i(k) + u'_i(k)Hx_i(k)\} + M_1 - M_2 \\
= & \sum_{i=1}^N \sum_{k=0}^{\infty} E\{[u_i(k) + G^\dagger Hx_i(k)]'G[u_i(k) + G^\dagger Hx_i(k)]\} + M_1 - M_2.
\end{aligned}$$

It can be verified that

$$\begin{aligned}
& \sum_{k=0}^{\infty} E\{[u_i(k) + G^\dagger Hx_i(k)]'G[u_i(k) + G^\dagger Hx_i(k)]\} \\
= & \sum_{k=0}^{\infty} E\left\{[u_i(k) + G^\dagger Hx_i(k) - (I - G^\dagger G)z_i(k)]'G[u_i(k) \right. \\
& \left. + G^\dagger Hx_i(k) - (I - G^\dagger G)z_i(k)]\right\}, \tag{23}
\end{aligned}$$

where $z_i(k)$ is an arbitrary vector with compatible dimension. Then, the optimal controller is given by the first equation of (22) because of the positive semi-definiteness of G . Substituting (22) into MAS (1), it yields that

$$x_i(k+1) = [\mathcal{A}x_i(k) + \mathcal{B}z_i(k)] + [\mathcal{C}x_i(k) + \mathcal{D}z_i(k)]w(k). \tag{24}$$

From Theorem 3.5, we have $\mathcal{P} = \mathcal{H}_\gamma(\mathcal{P})$, which has the form (3) with A, B, C, D being replaced by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, respectively.

Let $X(k) = [x'_1(k), \dots, x'_N(k)]'$, $\delta_i(k) = x_i(k) - \frac{1}{N} \sum_{i=1}^N x_i(k)$ ($i = 1, 2, \dots, N$) and $\delta(k) = [\delta'_1(k), \dots, \delta'_N(k)]'$. By the similar procedures of Theorem 4.1, (24) can be rewritten as

$$X(k+1) = [I_N \otimes \mathcal{A} - L \otimes (\mathcal{B}K)]X(k) + [I_N \otimes \mathcal{C} - L \otimes (\mathcal{D}K)]X(k)w(k).$$

We further obtain

$$\delta(k+1) = [I_N \otimes \mathcal{A} - L \otimes (\mathcal{B}K)]\delta(k) + [I_N \otimes \mathcal{C} - L \otimes (\mathcal{D}K)]\delta(k)w(k).$$

Select a unitary matrix $\Psi = \left[\frac{1}{\sqrt{N}}, \phi_2, \dots, \phi_N\right]$ such that $\Psi' L \Psi = \text{diag}\{0, \lambda_2, \dots, \lambda_N\}$, where ϕ_i satisfies $\phi'_i L = \lambda_i \phi'_i$ for $i = 2, \dots, N$. Let $\delta(k) = (\Psi \otimes I_n) \tilde{\delta}(k)$ and $\tilde{\delta}(k) = \left[\tilde{\delta}'_1(k), \dots, \tilde{\delta}'_N(k)\right]'$. It yields that $\tilde{\delta}_1(k) = 0$ and

$$\begin{aligned}
\tilde{\delta}(k+1) &= [I_N \otimes \mathcal{A} - \text{diag}\{0, \lambda_2 \mathcal{B}K, \dots, \lambda_N \times \mathcal{B}K\}]\tilde{\delta}(k) \\
&+ [I_N \otimes \mathcal{C} - \text{diag}\{0, \lambda_2 \mathcal{D}K, \dots, \lambda_N \mathcal{D}K\}]\tilde{\delta}(k)w(k).
\end{aligned}$$

Denote $\xi(k) = \left[\tilde{\delta}'_2(k), \dots, \tilde{\delta}'_N(k)\right]'$. Then, we have

$$\xi(k+1) = [I_{N-1} \otimes \mathcal{A} - \wedge \otimes \mathcal{B}K]\xi(k) + [I_{N-1} \otimes \mathcal{C} - \wedge \otimes \mathcal{D}K]\xi(k)w(k), \tag{25}$$

where $\wedge = \text{diag}\{\lambda_2, \dots, \lambda_N\}$.

Similarly, denote $V(k) = \xi'(k)(I_{N-1} \otimes \mathcal{P})\xi(k)$. Taking the mathematical expectation of $EV(k+1)$, we obtain

$$EV(k+1) = E\{\xi'(k)[\mathcal{R}'_1(I_{N-1}\mathcal{P})\mathcal{R}_1 + M'_1(I_{N-1} \otimes \mathcal{P})M_1]\xi(k)\},$$

and

$$\begin{aligned}
EV(k+1) - EV(k) &= E\{\xi'(k)[\mathcal{R}'_1(I_{N-1}\mathcal{P})\mathcal{R}_1 \\
&+ M'_1(I_{N-1} \otimes \mathcal{P})M_1 - I_{N-1} \otimes \mathcal{P}]\xi(k)\},
\end{aligned}$$

where $\mathcal{R}_1 = I_{N-1} \otimes \mathcal{A} - \wedge \otimes \mathcal{B}K$ and $\mathcal{M}_1 = I_{N-1} \otimes \mathcal{C} - \wedge \otimes \mathcal{D}K$. Note that (11) holds. It yields that

$$\mathcal{A}'\mathcal{P}\mathcal{A} + \mathcal{C}'\mathcal{P}\mathcal{C} - (2\lambda_i k - \lambda_i^2 k^2)(\mathcal{A}'\mathcal{P}\mathcal{B} + \mathcal{C}'\mathcal{P}\mathcal{D})$$

$$\begin{aligned}
& \times (\mathcal{D}'\mathcal{P}\mathcal{D} + \mathcal{B}'\mathcal{P}\mathcal{B})^\dagger (\mathcal{B}'\mathcal{P}\mathcal{A} + \mathcal{D}'\mathcal{P}\mathcal{C}) - \mathcal{P} \\
& < \mathcal{A}'\mathcal{P}\mathcal{A} + \mathcal{C}'\mathcal{P}\mathcal{C} - \gamma (\mathcal{A}'\mathcal{P}\mathcal{B} + \mathcal{C}'\mathcal{P}\mathcal{D}) \\
& \times (I_m + \mathcal{D}'\mathcal{P}\mathcal{D} + \mathcal{B}'\mathcal{P}\mathcal{B})^{-1} (\mathcal{B}'\mathcal{P}\mathcal{A} + \mathcal{D}'\mathcal{P}\mathcal{C}) + I_n - \mathcal{P} = 0.
\end{aligned}$$

Letting $\bar{K} = k (I_m + \mathcal{D}'\mathcal{P}\mathcal{D} + \mathcal{B}'\mathcal{P}\mathcal{B})^{-1} (\mathcal{B}'\mathcal{P}\mathcal{A} + \mathcal{D}'\mathcal{P}\mathcal{C})$, we can obtain

$$(\mathcal{A} - \lambda_i \mathcal{B}\bar{K})' \mathcal{P} (\mathcal{A} - \lambda_i \mathcal{B}\bar{K}) + (\mathcal{C} - \lambda_i \mathcal{D}\bar{K})' \mathcal{P} (\mathcal{C} - \lambda_i \mathcal{D}\bar{K}) - \mathcal{P} < 0,$$

which can be reformulated as

$$\begin{aligned}
\mathcal{R}_2 &= \text{diag} \left\{ (\mathcal{A} - \lambda_2 \mathcal{B}\bar{K})' \mathcal{P} (\mathcal{A} - \lambda_2 \mathcal{B}\bar{K}), \dots, \right. \\
& \quad \left. (\mathcal{A} - \lambda_N \mathcal{B}\bar{K})' \mathcal{P} (\mathcal{A} - \lambda_N \mathcal{B}\bar{K}) \right\} \\
&+ \text{diag} \left\{ (\mathcal{C} - \lambda_2 \mathcal{D}\bar{K})' \mathcal{P} (\mathcal{C} - \lambda_2 \mathcal{D}\bar{K}), \dots, \right. \\
& \quad \left. (\mathcal{C} - \lambda_N \mathcal{D}\bar{K})' \mathcal{P} (\mathcal{C} - \lambda_N \mathcal{D}\bar{K}) \right\} < 0.
\end{aligned}$$

Then, similar to the derivation process of Theorem 4.1, we can obtain that mean square consensus can be achieved for the MAS (1).

In addition, letting $z_i(k) = \frac{1}{N} \sum_{i=1}^N x_i(k)$, where $z(0) = \frac{1}{N} \sum_{i=1}^N x_i(0)$, we have

$$z(k+1) = \mathcal{A}y(k) + \mathcal{C}z(k)w(k).$$

Denote $\tilde{V}(k) = E[z'(k)Pz(k)] \geq 0$, where P satisfies ARE (21). Then, we obtain

$$\begin{aligned}
\tilde{V}(k+1) - \tilde{V}(k) &= E[z'(k) (\mathcal{A}'P\mathcal{A} + \mathcal{C}'P\mathcal{C} - P) z(k)] \\
&= E\{z'(k) [-Q - G'H^\dagger R H^\dagger G] z(k)\} \\
&\leq 0,
\end{aligned}$$

which implies

$$\lim_{k \rightarrow \infty} \tilde{V}(k) = \lim_{k \rightarrow \infty} E[z'(k)Pz(k)] = \theta,$$

where $\theta \geq 0$ is a constant. The consensus value of MAS (1) is given by (18), which shows that

$$\lim_{k \rightarrow \infty} E[x'_i(k)Px_i(k)] = \lim_{k \rightarrow \infty} E[z'(k)Pz(k)] = \theta.$$

Thus, the optimal value is given by

$$\sum_{i=1}^N J_i = \sum_{i=1}^N E[x'_i(0)Px_i(0)] - \theta N.$$

□

6. Simulation example. Consider the MAS (1) and the cost function (2) with A, B, C, D, Q, R as follows:

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2.5 & 0 \\ 0 & 0.5 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \\
D &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

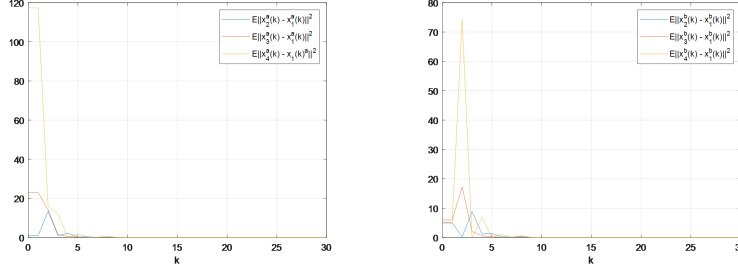


FIGURE 1. The trajectory of $E||x_i(k) - x_1(k)||^2$ and $E||y_i(k) - y_1(k)||^2$, $i = 2, 3, 4$.

which satisfy $\text{Range}(H) \subseteq \text{Range}(G)$. Consider $\mathcal{G} = \{\mathcal{V}, \mathcal{M}\}$, where $\mathcal{V} = \{1, 2, 3, 4\}$ and

$$\mathcal{M} = [a_{ij}]_{4 \times 4} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Further we obtain the Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

and its eigenvalues: $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 4$. Let initial values $x_1(0) = [5.59 \ 1.16]'$, $x_2(0) = [4.57 \ -1.10]'$, $x_3(0) = [0.79 \ -1.28]'$, $x_4(0) = [-5.24 \ -1.04]'$.

Letting $\gamma = 0.86 > \gamma^* = 0.75$, one can calculate that $\mathcal{P} = \begin{bmatrix} 15.1473 & 0 \\ 0 & 1.3333 \end{bmatrix}$,

$P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ by solving (3) and (21). We have

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 2.5 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, the condition of Theorem 5.1 is satisfied and the optimal consensus control is

$$u_i(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} K \sum_{j=1}^4 a_{ij} (x_j(k) - x_i(k))$$

with $K = k \begin{bmatrix} 0.7916 & 0 \\ 0 & 0 \end{bmatrix}$, where $k \in [0.29, 0.35]$. Denote $x_i^a(k)$, $x_i^b(k)$ as the first and second components of $x_i(k)$, respectively. As shown in Figure 1, Figure 2, Figure 3 and Figure 4, the mean square consensus and almost sure consensus of the system are achieved.

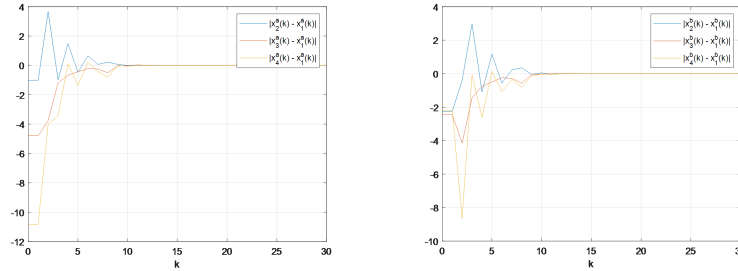


FIGURE 2. The trajectory of $|x_i(k) - x_1(k)|$ and $|y_i(k) - y_1(k)|$, $i = 2, 3, 4$.

7. Conclusions. In this work, we consider the optimal consensus control of discrete-time MASs with multiplicative noise. Based on two Riccati equations, the sufficient condition for optimal consensus are given respectively. In our future research, we are going to investigate the optimal consensus corrupted by both communication latency and multiplicative noise simultaneously. The coexistence of additive and multiplicative noise will also be studied. The optimal control with unknown mean and variance of multiplicative noise is also an interesting problem [30].

Acknowledgments. This work was supported by Funds for international Cooperation and Exchange of the National Natural Science Foundation of China under Grant 62261136550, National Natural Science Foundation of China under Grant 62073305, the Hubei Provincial Natural Science Foundation under Grant 2022CFA041, the National Natural Science Foundation of China under Grant T2293770.

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